

(1)

AD-A229 795

NONLINEAR WAVE AND DIFFUSION EQUATIONS

Final Technical Report

by

J.B. McLeod & J.R. Ockendon

August 1990

United States Army

EUROPEAN RESEARCH OFFICE OF THE US ARMY

London, England.

CONTRACT NUMBER DAJA45-86-C-0040

University of Oxford

Approved for Public Release; distribution unlimited

DTIC
ELECTE
DEC 04 1990
S D D

40 12 3

PROBLEMS STUDIED

The research undertaken under the present grant has consisted of a number of investigations into nonlinear wave and diffusion equations. These make their appearance in many mathematical modelling problems, and we have been interested in applications in the field of fluid dynamics, heat and mass transfer, and superconductivity. The problems were tackled using the expertise available both through permanent staff members and through a visitor programme, and equal emphasis was laid on analytical techniques, numerical techniques, and modelling.

SPECIFIC RESULTS

(1) Lens and Antenna design

The basic problem here (originally stimulated by a query from the United States Navy) is, given two points, to design an optical lens which will have the property that it focuses all rays of light from the one point onto the other. The problem reduces to the study of an unusual pair of functional-differential equations, and as part of research under the grant it was shown that if the lens is symmetric, i.e. if the two faces of the lens are the same, then there exist precisely two lenses which will give the required focusing, one in which the rays of light do not cross the axis of the lens, and the other in which they do.

There are many extensions of this problem, for example to lenses which are not symmetric or to lenses with variable refractive index. Investigation into these extensions has been a continuing part of the research programme. One general aspect of which has been studied intensively is the question of how well a slightly imperfect lens can perform. In several situations, there are "best possible" lenses and these include off-axis or axially displaced object points, and lenses focusing at two different wavelengths.



A-11

(2) Blow-up and Quenching for Nonlinear Diffusion Equations

Equations of the general type

$$u_t = \Delta u + f(u) \quad (1)$$

can exhibit blow-up solutions (when, at some finite time T , $u(x,T)$ becomes infinite for some value of x), or quenching of solutions (when at some finite time T a positive solution becomes identically zero). One of the most basic questions to be answered is whether, if blow-up does occur, it occurs at just one value of x (or perhaps a finite number of values of x) or alternatively at all x simultaneously. As part of the research under the grant, it was shown that, under very general conditions, at least in one dimension, blow-up does occur at most at a finite number of points, and this leads to a number of supplementary questions which have also been looked at under the programme. For example, what is the nature of the profile of the solution just before blow-up or quenching? What is the effect of introducing spatially-dependent coefficients in (1)? Does the one-dimensional result extend to higher dimensions? At least partial answers to the first two questions have been given. The third question remains open.

(3) Wedge Entry

This is the problem of the entry of a wedge into water, and models the impact of seas on a ship's hull. The investigation arose because of a query by the UK Navy. Although the governing equations are almost classical, little effective work had been done on them beyond numerical calculations. As part of the research programme we have been able to give a complete account of the existence and qualitative behaviour of the solutions, including an analysis of the asymptotic behaviour as the semi-angle of the wedge approaches π , i.e. as the wedge deadrise angle becomes very small, which is the important case for naval applications. Additional work on the asymptotic behaviour of slamming bodies of more general cross section has also been carried out by using a "displacement potential" to reduce the problem to a variational inequality.

(4) Asymptotics Beyond All Orders,

In quite a number of recent problems situations have arisen where asymptotic methods (say as a parameter $\epsilon \rightarrow 0$) give the wrong answer if only algebraic terms in ϵ are considered, and when it is necessary therefore to take account of terms of order $e^{-1/\epsilon}$. Two such instances are an equation arising in dendrite growth,

$$\epsilon \ddot{\theta} + \dot{\theta} = \cos \theta,$$

and an equation arising in wave propagation,

$$\epsilon \ddot{v} + \dot{v} = 1 - v^2.$$

A systematic investigation of such problems has been carried out, involving both analytical and numerical techniques.

(5) Higher-order Diffusion,

Problems involving the spread of viscous droplets, and also certain modelling problems in semiconductors, lead to higher-order diffusion equations of the type

$$\frac{\partial u}{\partial t} + (-1)^{m+1} \frac{\partial}{\partial x} (|u|^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}}) = 0. \quad (2)$$

The droplet case is the case $m = 1$, and the semiconductor $m = 2$. We have carried out an extensive investigation of the existence, uniqueness and qualitative behaviour of similarity solutions of (2) in the particular case $m = 1$ and are extending the work to other values of m .

(6) Macroscopic Models for Solidification,

The mathematical regularisation of ill-posed Stefan models for supercooled solidification has been studied from several aspects. The introduction of surface tension poses an interesting theoretical challenge and several conjectures have been made concerning the resulting morphology. A regularisation is also possible using a "mathematical" nucleation to create a new phase boundary in such a way that the well-posedness of the model is ensured. Yet another possibility is the introduction of kinetic undercooling at the free boundary and this powerful mechanism has been studied analytically and

asymptotically.

Certain explicit solutions to two-dimensional classical Stefan models have been catalogued and an asymptotic analysis of a simple continuous casting machine has been possible in the typical case when the liquid steel from the tundish forms a long thin finger in the caster.

As usual, members of the faculty and students from Oxford have participated in the triennial international free boundary conferences, most recently held in Irsee, Bavaria (June 1987) and Montreal (June 1990).

→ (7) Mathematical Models for Fibre Tapering

A simple hyperbolic model for the unsteady axial tapering of cylindrical optical fibre has been proposed and analysed to show that breakage is only possible in exceptional circumstances. This work has spurred further research into tapering of non-cylindrical fibres, and in particular the problem of core displacement when several fibres are being spliced by using the tapering mechanism.

→ (8) Semiconductor Fabrication

The derivation and study of hierarchies of nonlinear diffusion equations which arise in modelling semiconductor dopant diffusion were initiated during the early part of the contract and research on them has continued at other universities ever since. The most recent emphasis has been on nonlocal effects resulting in pairing of vacancies and dopant atoms: this results in the appearance of effective diffusion coefficients which depend on the boundary data.

→ (9) Superconductivity Modelling

Work has been initiated on the analysis of macroscopic models for phase transitions in superconductors. The Ginzburg-Landau theory (based on the BCS theory) leads to a complicated system of nonlinear diffusion equations for the magnetic field and the complex order parameter which describes the extent to which the transition has been completed. It appears that in various

asymptotic limits this system can be reduced either to vector Stefan problems or to transition layer problems for nonlinear ordinary differential equations, depending on the observer's length scale. The parameters appearing in these reduced models characterise both the "type" of the superconductor and the stability of the intermediate state.

(10) Hypersonic Flow

A study has been made of the hyperbolic system describing gas flow in the inviscid shock layers which occur in the vicinity of pistons and blunt bodies when the free stream Mach number is large and the specific heat ratio near unity. There are several kinds of piston motions for which no solution appears to exist and these situations have been studied using a regularisation technique which also sheds light on the interesting phenomenon of shock layer separation when the piston is abruptly stopped or when the blunt body suffers an abrupt change of geometry.

(11) Combustion in Porous Media

Early work on porous medium combustion dominated by excess gaseous fuel has been modified and extended to combustion in the presence of smaller amounts of gaseous fuel. The type of stability for these burning zones terminated by temperature or combustible solid exhaustion has been determined.

Publications

1. A. Friedman & J.B. McLeod, The optimal design of an optical lens, Arch. Rational Mech. Anal. 99 (1987), 147-164.
2. A. Friedman & J.B. McLeod, A lens that focuses two pairs of points, Arch. Rational Mech. Anal. 101 (1988), 57-83.
3. B. Van-Brunt, The Herschel and Abbe-Sine conditions as limiting cases in lens design, IMACS Conference, Paris (1988).
4. B. Van-Brunt, Functional-differential equations and lens design in geometrical optics, D.Phil Thesis (Oxford) (1989).
5. B. Van-Brunt & J.R. Ockendon, A lens focusing light at two different wavelengths, Submitted to J. Math. Anal. Applies. (1990). (to appear)
6. A. Friedman & J.B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana U. Math. J. 34 (1985), 425-447.
7. A. Friedman & J.B. McLeod, Blow-up of solutions of nonlinear degenerate parabolic equations, Arch. Rational Mech. Anal. 96 (1986), 55-80.
8. A. Friedman, J. Friedman & J.B. McLeod, Concavity of solutions of nonlinear ordinary differential equations, J. Math. Anal. Applies. 131 (1988), 486-500.
9. M.S. Floater, Blow-up at the boundary for nonlinear diffusion equations, Arch. Rational Mech. Anal., to appear. (1990)
10. S.K. Wilson, J.R. Ockendon, S.D. Howison, Incompressible Water Entry Problems at Small deadrise angles, (1990) J. Fluid Mech., to appear.
11. C.J. Amick & J.B. McLeod, A singular perturbation problem in needle crystals, Arch. Rational Mech. Anal. 109 (1990), 139-171.
12. C.J. Amick & J.B. McLeod, A singular problem in water waves, Stability and Applied Analysis of Continuous Media, to appear.
13. J.M. Hammersley & G. Mazzarino, A differential equation connected with the dendritic growth of crystals, IMA J. App. Math. (1989) 42 43-57.
14. F. Bernis & J.B. McLeod, Similarity solutions of a higher-order nonlinear diffusion equation, Nonlinear Analysis, (to appear).
15. F. Bernis, L.A. Peletier & S.M. Williams, Source type solutions of a fourth order nonlinear degenerate parabolic equation, preprint.
16. S.D. Howison, A.A. Lacey & J.R. Ockendon, Hele-Shaw free boundary problems with suction, Q.J. Mech. App. Math. 1988 41 183-193.
17. S.D. Howison, A.A. Lacey & J.R. Ockendon, Irregular morphologies in unstable Hele-Shaw free boundary problems. Q.J. Mech. App. Math. 43 387-405.
18. A. Fasano, S.D. Howison, J.R. Ockendon & M. Primicerio. Some remarks on the regularisation of the supercooled one-phase Stefan problem. Quart. App. Math. 1990.

19. J.N. Dewynne, S.D. Howison, J.R. Ockendon & W. Xie, Asymptotic behaviour of solutions of Stefan problems with kinetic undercooling. *J. Austr. Math. Soc. B* 1989 31 81-96.
20. W. Xie, The Stefan problem with a kinetic condition at the free boundary, *SIAM J. Math. Anal.* 1990 (to appear).
21. S.D. Howison, Similarity solutions to the Stefan problem and the binary alloy problem, *IMA J. App. Math.* 1988 40 141-162.
22. S.D. Howison, J.R. King, Explicit solution to six free boundary problems, *IMA J. App. Math.* 1989.
23. J.N. Dewynne, J.R. Ockendon, Numerical and analytical solution of continuous casting problems. *Proc. Oberwolfach conference on free boundaries and optimal control*. Berkhauser Verlag 1990. (eds. K-H Hoffmann, J. Sprekels).
24. Irsee Conference: Free Boundary Problems; Theory and Application. Pitman Res. Notes in Maths, 1987 (eds. K.H. Hoffmann, J. Sprekels).
25. Montreal Conference: Free Boundary Problems; Theory and Application. Pitman Res. Notes in Maths, 1990 (eds. J. Chadam, H. Rasmussen).
26. J.R. King, C. Please, Diffusion of dopant in crystalline silicon. *IMA J. App. Math.* (1987) 37 185-198.
27. J.R. King, Asymptotic Analysis of an Impurity-defect model for dopant diffusion. *Q. J. Mech. App. Math.* 1991 (to appear)
28. S.J. Chapman, S.D. Howison, J.B. McLeod, J.R. Ockendon, Normal-Superconductivity transitions in Ginzburg-Landau theory. Submitted to *Proc. Roy. Soc. Edin.* (1990).
29. K. Louie, 1990, D.Phil thesis, Oxford University.
30. K. Louie & J.R. Ockendon, Mathematical aspects of the theory of inviscid hypersonic flow. *Phil. Trans. Royal Soc. Lond.* 1991 (to appear).
31. J. Norbury & A. M. Stuart, Travelling combustion waves in a porous medium. Part I - Existence. *SIAM J. App. Math.* (1988) 48 155-169.
32. J. Norbury & A.M. Stuart, Travelling combustion waves in a porous medium. Part II - Stability. *SIAM J. App. Math.* (1988) 48 374-392.
33. J. Norbury & A.M. Stuart, A model for porous-medium combustion. *Q. J. Mech. App. Math.* 42 1 (1989).

Personnel

Permanent Staff:

Dr J.M. Hammersley
Dr S.D. Howison
Dr J.B. McLeod
Dr J. Norbury
Dr H. Ockendon
Dr J.R. Ockendon
Dr A.B. Tayler

Visitors

Prof C.J. Amick (Chicago)
Prof F. Bernis (Barcelona)
Prof J. Chadam (McMaster)
Prof L.E. Fraenkel (Bath)
Prof. A Friedman (Minneapolis)
Dr A.K. Head (Melbourne)
Prof M.A. Herrero (Madrid)
Prof L.A. Peletier (Leiden)
Prof J.L. Vazquez (Madrid)

Research Assistants

M.S. Floater	(D.Phil 88)
J.R. King	(D.Phil 86)
I.M. Lee	(D.Phil 88)
C.J. Mabb	(D.Phil 87)
B. Van-Brunt	(D.Phil 89)
S.M. Williams	(D.Phil 89)
S.K. Wilson	(D.Phil 89)
S.J. Chapman	(M.Sc. 90)
K. Louie	(D.Phil 90)